XII. *A Conjecture concerning the Method by which* Cardan's *Rules for resolving the Cubic Equation* $x^3+qx=r$ *in all cases (or in all magnitudes of the known quantities q and r) and the Cubic Equation* $x^3-qx=r$ *in the first Case of it (or when r is greater than* $\frac{2q\sqrt{q}}{3\sqrt{3}}$, *or* $\frac{rr}{4}$ *is greater than* $\frac{q^3}{27}$) *were probably discovered by* Scipio Ferreus, *of* Bononia, *or whoever else was the first Inventor of them. By* Francis Maseres, *Esq. F. R. S. Cursitor Baron of the Exchequer.*

Read January 27, 1780.

ARTICLE I.

THERE is nothing more amusing, or more grateful to an inquisitive mind, in the study of the sciences of Geometry and Algebra (for if we banish from it the ridiculous mysteries arising from the supposition of *negative* quantities, or quantities *less than nothing*, the latter may deserve the name of a *science* as well as the former) than to contemplate the methods by which the several ingenious and surprizing truths that are delivered in the books

books that treat of them were firſt diſcovered. This we are ſometimes enabled to do by the authors themſelves to whom we are indebted for theſe diſcoveries, who have candidly informed their readers of the ſeveral ſteps, and ſometimes of the accidents, by which they have been led to them: but it alſo often happens, that the authors of theſe diſcoveries have neglected to give their readers this ſatiſfaction, and have contented themſelves with either barely delivering the propoſitions they have found out, without any demonſtrations, or with giving formal and poſitive demonſtrations of them, which command indeed the aſſent of the underſtanding to their truth, but afford no clue whereby to diſcover the train of reaſoning by which they were firſt found out; and conſequently contribute but little to enable the reader to make ſimilar diſcoveries himſelf on the like ſubjects. This ſeems to be the caſe with thoſe ingenious rules for the reſolution of certain cubic equations, which are uſually known by the name of CARDAN's rules. We are told to make certain ſubſtitutions of ſome quantities for others in theſe equations $x^3+qx=r$ and $x^3-qx=r$ (which are the objects of thoſe rules) and certain ſuppoſitions concerning the quantities ſo ſubſtituted; by doing which we find, that thoſe equations will be transformed into other equations which will involve the ſixth power of the unknown quantity

tity contained in them, but which (though of double the dimenſions of the original equations $x^3+qx=r$ and $x^3-qx=r$, from which they were derived) will be more eaſy to reſolve than thoſe equations, becauſe they will contain only the ſixth power and the cube of the unknown quantity which is their root, and conſequently will be of the ſame form as quadratic equations; ſo that by reſolving them as quadratic equations we may obtain the value of the cube of the unknown quantity which is their root, and afterwards, by extracting the cube-root of the ſaid value, we may obtain the value of the ſaid root, or unknown quantity, itſelf; and then at laſt, by the relation of this laſt root to x, or the root of the original equation, (which relation is derived from the ſuppoſitions that have been made in the courſe of the preceding tranſformations) we may determine the value of x. And, if we pleaſe to examine the ſeveral ſteps of this proceſs with ſufficient attention, we may perceive, as we go along, that all theſe ſubſtitutions are legitimate and practicable, or are founded upon poſſible ſuppoſitions; though I cannot but obſerve, that the writers on algebra, for the moſt part, have not been ſo kind as to ſhew us that they are ſo. But ſtill the queſtion recurs, "How came SCIPIO "FERREUS, of Bononia (who, as CARDAN tells us, was "the firſt inventor of theſe rules) or the other perſon,

"whoever

" whoever he was, that invented them, to think of mak-
" ing these lucky substitutions which thus transform the
" original cubic equations into equations of the sixth
" power which contain only the sixth and third powers
" of the unknown quantities which are their roots, and
" consequently are of the form of quadratic equations?"
To answer this question as well as I can *by conjecture* (for I know of no historical account of this matter in any book of algebra) and in a manner that appears to me to be *probable*, is the design of the following pages.

2. The most probable conjecture concerning the invention of these rules, called CARDAN's rules, by SCIPIO FERREUS, of Bononia, or whoever else was the inventor of them, seems to be this: that the said inventor tried a great variety of methods of reducing the three cubic equations of the third class, to wit, $x^3+qx=r$ and $x^3-qx=r$, and $qx-x^3=r$ (to some one of which all other cubic equations may, by proper substitutions, be reduced) to a lower degree, or to a more simple form, by substituting various quantities in the stead of x, in hopes that some of the terms arising by such substitutions might be equal to others of them, and, having contrary signs prefixed to them, might destroy them, and thereby render the new equation more simple and manageable than the old one. And, amongst other trials, it seems natural to

imagine, that he would fubftitute the fum or difference of two other quantities inftead of x, as being the moft fimple and obvious fubftitutions that could be made. And by making thefe fubftitutions, the above mentioned rules would of courfe come to be difcovered, as well as the aforefaid limitation of them in the refolution of the equation $x^3 - qx = r$, which reftrains the rule to thofe cafes only in which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, and their utter inutility in all the cafes of the equation $qx - x^3 = r$. This will appear by examining each of thefe equations feparately in the following manner.

Of the equation $x^3 + qx = r$.

ART. 3. In the equation $x^3 + qx = r$ the inveftigator of thefe rules would naturally be inclined to fubftitute the difference of two quantities (which we will here call y and z, and of which we will fuppofe y to be the greater) inftead of x, rather than their fum, or would fuppofe x to be equal to $y - z$, rather than to $y + z$; becaufe, if he was to fuppofe x to be equal to the fum of the two quantities y and z, and was to fubftitute that fum, or the binomial quantity $y + z$, inftead of x in the equation $x^3 + qx = r$, it is evident, that (as the figns of x^3 and qx

 are,

are, both of them, affirmative) the terms of the new equation, arifing from fuch fubftitution, would all of them be likewife affirmative; and confequently none of them, though they fhould happen to be exactly equal to each other, could exterminate each other, and thereby render the new equation more fimple than the old one, which was the only view with which the fubftitution would have been made. He would, therefore, fuppofe x to be equal to $y-z$; and by fubftituting this quantity inftead of x in the original equation $x^3+qx=r$, he would transform that equation into the following one, to wit,

$$y^3-3yyz+3yzz-z^3+qy-qz=r,$$

$$\text{or } y^3-3yz\times\overline{y-z}-z^3+q\times\overline{y-z}=r.$$

Now in this equation it is evident, that the terms $3yz\times\overline{y-z}$ and $q\times\overline{y-z}$ have contrary figns; and therefore, if their co-efficients $3yz$ and q can be fuppofed to be equal to each other, thofe terms will mutually deftroy each other, and the equation will be reduced to the following fhort one, $y^3-z^3=r$. And if in this equation we fubftitute, inftead of z, its value $\frac{q}{3y}$, derived from the fame fuppofition of the equality of q and $3yz$, the equation will be $y^3-\frac{q^3}{27y^3}=r$; and, by multiplying both fides by y^3, it will be $y^6-\frac{q^3}{27}=ry^3$; which equation, though it rifes to the fixth power of the unknown quantity y, is

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evidently of the form of a quadratic equation, and may therefore be refolved, fo far as to find the value of the cube of y, in the fame manner as a quadratic equation; after which it will be poffible to find the value of y itfelf by the mere extraction of the cube root; and then at laft, from the relation of y to x (derived from the foregoing fuppofitions that $y-z$ was equal to x, and that $3yz$ was equal to q, and confequently z equal to $\frac{q}{3y}$) we fhall be able to determine the value of x.

Art. 4. It would therefore remain for the inveftigator of this method to inquire, whether or no the fuppofition, " that $3yz$ was equal to q," was a poffible fuppofition; that is, whether it was poffible (whatever might be the magnitudes of q and r) for two quantities, y and z, to exift, whofe nature fhould be fuch that their difference $y-z$ fhould be equal to the unknown quantity x in the equation $x^3+qx=r$, and that three times their product fhould at the fame time be equal to q, or their fimple product to the third part of q. And this fuppofition he would foon find to be always poffible, whatever may be the magnitudes of q and r; becaufe, if the leffer quantity z is fuppofed to increafe from *o ad infinitum*, and the greater quantity y is likewife fuppofed to increafe with equal fwiftnefs, or to receive equal increments in the fame

 times,

times, and thereby to preferve their difference $y-z$ always of the fame magnitude, or equal to x, it is evident that the product or rectangle yz will increafe continually at the fame time from *o ad infinitum*, and confequently will pafs fucceffively through all degrees of magnitude, and therefore muft at one point of time during its increafe become equal to $\frac{q}{3}$.

And having thus found this fuppofition of the equality of yz and $\frac{q}{3}$, or of $3yz$ and q, to be always poffible, whatever might be the magnitudes of q and r, our inveftigator would juftly confider his folution of the equation $x^3+qx=r$ (which was founded on that fuppofition) as legitimate and compleat. And thus we fee in what manner it feems probable, that CARDAN's rule for refolving the cubic equation $x^3+qx=r$ may have been difcovered.

Of the equation $x^3-qx=r$.

Art. 5. In this fecond equation $x^3-qx=r$, in which the fecond term qx is fubtracted from the firft, or marked with the fign $-$, it feems to have been natural for the perfon who invented thefe rules to fubftitute the *fum* as well as the *difference* of two other quantities, y and z, inftead

ftead of x, in the terms x^3 and qx, in hopes of fuch an extermination of equal terms, and confequential reduction of the equation to one of a fimpler and more manageable form, as was found to be fo ufeful in the cafe of the former equation $x^3+qx=r$. We will therefore try both thefe fubftitutions; and, as that of the difference $y-z$ has in the former cafe proved fo fuccefsful, we will begin by that.

Art. 6. Now, by fubftituting the difference $y-z$ inftead of x in the equation $x^3-qx=r$, we fhall transform it into the following equation, to wit, $y^3-3yyz+3yzz-z^3-q\times\overline{y-z}|=r$, or $y^3-3yz\times\overline{y-z}-z^3-q\times\overline{y-z}=r$; in which the terms $3yz\times\overline{y-z}$ and $q\times\overline{y-z}$ have both of them the fame fign $-$ prefixed to them, and confequently can never exterminate each other, whether $3yz$ be equal or unequal to q. This fubftitution therefore is in this cafe of no ufe.

Art. 7. We will now therefore try the fubftitution of the *fum* of y and z, inftead of their difference, in the equation $x^3-qx=r$.

Now, if x be fuppofed to be equal to $y+z$, and $y+z$ be fubftituted inftead of it in the equation $x^3-qx=r$, that equation will be thereby transformed into the following one, to wit,

y^3+

$$y^3+3yyz+3yzz+z^3-q\times\overline{y+z}=r,$$

$$\text{or } y^3+3yz\times\overline{y+z}+z^3-q\times\overline{y+z}=r.$$

Now in this equation, the terms $3yz\times\overline{y+z}$ and $q\times\overline{y+z}$ have contrary ſigns. Conſequently, if they can be ſuppoſed to be equal to each other, they will deſtroy each other, and the equation will be thereby reduced to the following ſhort one, $y^3+z^3=r$; that is, if $3yz$ and q can be ſuppoſed to be equal to each other, or if yz can be ſuppoſed to be equal to $\frac{q}{3}$, the equation will be reduced to the ſhort equation $y^3+z^3=r$. And, if in this ſhort equation we ſubſtitute, inſtead of z, its value $\frac{q}{3y}$ (derived from the ſame ſuppoſition of the equality of $3yz$ and q) the equation thence reſulting will be $y^3+\frac{q^3}{27y^3}=r$; and by multiplying both ſides by y^3, it will be $y^6+\frac{q^3}{27}=ry^3$; and, by ſubtracting y^6 from both ſides, it will be $ry^3-y^6=\frac{q^3}{27}$; which, though it riſes to the ſixth power of y, is evidently of the form of a quadratic equation, and conſequently may be reſolved in the ſame manner as a quadratic equation, ſo far as to find the value of y^3, or the cube of the root y; after which it will be poſſible to find the value of y itſelf by the mere extraction of the cube root; and, laſtly, from the relation of y to x (contained in the two ſuppoſitions, that $y+z$ is equal to x, and that

3yz

$3yz$ is equal to q, and consequently that z is equal to $\frac{q}{3y}$) we may determine the value of x.

Art. 8. The only thing, therefore, that would remain for the investigator of these rules to do, in order to know whether the foregoing method of resolving the equation $x^3+qx=r$ was practicable or not, would be to inquire, whether it was possible in all cases, that is, in all magnitudes of the known quantities q and r, for $3yz$ to be equal to q, or for yz (or the product or rectangle of the two quantities y and z, whose sum is equal to x) to be equal to $\frac{q}{3}$, and, if it was not possible in all cases, but only in some, to determine in what cases it *was* possible, or what must be the relation between q and r to make it possible.

Art. 9. Now, in order to determine this question, it would be proper and natural to observe, that the quantity yz, or the product of the two quantities y and z, whose sum is supposed to be equal to x, can never be greater than the square of half that sum, that is, than the square of $\frac{x}{2}$, or than $\frac{xx}{4}$, by El. 2, 5, but may be of any magnitude that does not exceed that square. Therefore, if $\frac{q}{3}$ is greater than $\frac{xx}{4}$, it will be impossible for yz to be equal to it; but, if $\frac{q}{3}$ is either equal to, or less than, $\frac{xx}{4}$,

$\frac{xx}{4}$, it will be poſſible for yx to be equal to it, and if $\frac{q}{3}$ is exactly equal to $\frac{xx}{4}$, z will be exactly equal to y, and each of them equal to one half of x. We muſt, therefore, inquire what is the magnitude of x when $\frac{q}{3}$ is equal to $\frac{xx}{4}$. Now, when $\frac{xx}{4}$ is $= \frac{q}{3}$, xx will be $= \frac{4q}{3}$, and $x = \frac{2\sqrt{q}}{\sqrt{3}}$: therefore, when x is leſs than $\frac{2\sqrt{q}}{\sqrt{3}}$, it will be impoſſible for yz to be equal to $\frac{q}{3}$; but when x is greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, it will be poſſible for yz to be equal to $\frac{q}{3}$.

But when x is $= \frac{2\sqrt{q}}{\sqrt{3}}$, x^3 will be $= \frac{8q\sqrt{q}}{3\sqrt{3}}$, and qx will be $= \frac{2q\sqrt{q}}{\sqrt{3}}$ or $\frac{6q\sqrt{q}}{3\sqrt{3}}$, and conſequently $x^3 - qx$ will be $= \frac{8q\sqrt{q}}{3\sqrt{3}} - \frac{6q\sqrt{q}}{3\sqrt{3}} = \frac{2q\sqrt{q}}{3\sqrt{3}}$.

Therefore, if it be true (as we ſhall preſently ſee that it is) that while x increaſes from being equal to $\sqrt{q}$ (which is evidently its leaſt poſſible magnitude) to any other magnitude, the compound quantity $x^3 - qx$, or the exceſs of x^3 above qx, will alſo continually increaſe from o (to which it is equal when x is $= \sqrt{q}$, or xx is $= q$) to ſome correſpondent magnitude without ever decreaſing; it will follow that, when x is leſs than $\frac{2\sqrt{q}}{\sqrt{3}}$, the compound quantity $x^3 - qx$ will be leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$; and when

x is

x is greater than $\frac{2\sqrt{q}}{\sqrt{q}}$, the compound quantity x^3-qx will be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$; and, *è converso*, if the compound quantity x^3-qx is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, x will be less than $\frac{2\sqrt{q}}{\sqrt{3}}$; and, if the compound quantity x^3-qx is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, x will be greater than $\frac{2\sqrt{q}}{\sqrt{3}}$. Consequently, if the compound quantity x^3-qx, or, its equal, the absolute term r in the equation $x^3-qx=r$, is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, it will be impossible for yz to be equal to $\frac{q}{3}$; but, if x^3-qx, or r, is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it will be possible for yz to be equal to $\frac{q}{3}$. Therefore, if x^3-qx, or r, is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, the foregoing method of resolving the cubic equation $x^3-qx=r$ will be impracticable; but, if $x^3-qx=r$, or r, is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it will be practicable.

Art. 10. It now only remains to be proved, that while x increases, from being equal to $\sqrt{q}$, *ad infinitum*, the compound quantity x^3-qx will likewise increase from o *ad infinitum*, without ever decreasing. Now this may be demonstrated as follows.

Art. 11. It is evident, that while x increafes from being equal to $\sqrt{q}$ *ad infinitum*, both the quantities x^3 and qx will increafe *ad infinitum* likewife. But it does not therefore follow, that the excefs of x^3 above qx will continually increafe at the fame time. This will depend upon the relation of the contemporary increments of x^3 and qx; if the increment of x^3 in any given time is equal to the contemporary increment of qx, the compound quantity x^3-qx will neither increafe nor decreafe, but continue always of the fame magnitude during the faid time, notwithftanding the increafe of x; if the former increment is lefs than the latter, the faid compound quantity will decreafe; and, if it is greater, it will increafe. We muft therefore inquire, whether the increment of x^3 in any given time is greater or lefs than the contemporary increment of qx.

Art. 12. Now, if $\dot{x}$ be put for the increment which x receives in any given time, the increment of x^3 in the fame time will be the excefs of $\overline{x+\dot{x}}|^3$ above x^3, that is, the excefs of $x^3+3x^2\dot{x}+3x\dot{x}^2+\dot{x}^3$ above x^3; and the increment of qx in the fame time will be the excefs of $q\times\overline{x+\dot{x}}$, or $qx+q\dot{x}$, above qx; that is, the increment of x^3 will be $3x^2\dot{x}+3x\dot{x}^2+\dot{x}^3$, and that of qx will be $q\dot{x}$. Now in the equation $x^3-qx=r$ it is evident, that xx muft be greater than q; for otherwife x^3 would not be greater than qx,

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as it is fuppofed to be. Confequently, $x\dot{x} \times \dot{x}$ muft be greater than $q\dot{x}$; and, *à fortiori,* $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ (which is more than triple of $x^2\dot{x}$) muft be greater than $q\dot{x}$; that is, the increment of x^3 will be greater than the contemporary increment of qx. Therefore, the excefs of x^3 above qx, or the compound quantity $x^3 - qx$, will increafe continually, without decreafing, while x increafes from $\sqrt{q}$ *ad infinitum.* Q. E. D.

Art. 13. It follows, therefore, upon the whole of thefe inquiries, that if the compound quantity $x^3 - qx$, or, its equal, the abfolute term r, is lefs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is lefs than $\frac{q^3}{27}$, it will be impoffible for yz to be equal to $\frac{q}{3}$, and confequently the foregoing method of refolving the equation $x^3 - qx = r$ will be impracticable; but, if $x^3 - qx$ or r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it will be poffible for yz to be equal to $\frac{q}{3}$, and confequently, the foregoing method of refolving the equation $x^3 - qx = r$ will be practicable. And thus we fee in what manner it is probable that CARDAN'S rule for refolving the cubic equation $x^3 - qx = r$ in the firft cafe of it, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, together with the reftriction of it to that firft cafe, may have been difcovered.

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Of the Equation $qx-x^3=r$.

Art. 14. In the third equation $qx-x^3=r$ the terms x^3 and qx have different ſigns, as well as in the ſecond equation $x^3-qx=r$; and therefore it ſeems to have been natural for the inventor of CARDAN's rules to try both the ſubſtitutions of $y-z$ and $y+z$ inſtead of x in this equation, as well as in that ſecond equation, in hopes of an extermination of equal terms that are marked with contrary ſigns, and a conſequent reduction of the equation to another which, though of double the dimenſions of the equation $qx-x^3=r$, ſhould have been of a ſimpler form and more eaſy to be reſolved. But it will be found upon trial, that neither of theſe ſubſtitutions will anſwer the end propoſed.

Art. 15. For, in the firſt place, let us ſuppoſe x to be $=y-z$. Then we ſhall have $x^3=y^3-3yyz+3yyz-z^3=y^3-3yz\times\overline{y-z}-z^3$, and $qx=q\times\overline{y-z}$, and conſequently $qx-x^3=q\times\overline{y-z}-y^3+3yz\times\overline{y-z}+z^3$. Therefore, $q\times\overline{y-z}-y^3+3yz\times\overline{y-z}+z^3$ will be $=r$. Now in this equation it is evident, the terms $q\times\overline{y-z}$ and $3yz\times\overline{y-z}$ have the ſame ſigns, and therefore can never deſtroy each other. Therefore, no ſuch method of reſolving this equation

$qx-x^3=r$ as was found above for the two former equations $x^3+qx=r$ and $x^3-qx=r$, can be obtained by ſubſtituting the difference $y-z$ in it inſtead of x.

Art. 16. We will now try the ſubſtitution of $y+z$ inſtead of x in the terms of this equation.

Now, if x be ſuppoſed to be $=y+z$, we ſhall have $x^3=y^3+3yyz+3yzz+z^3=y^3+3yz\times\overline{y+z}+z^3$, and $qx=q\times\overline{y+z}$, and conſequently, $qx-x^3=q\times\overline{y+z}-y^3-3yz\times\overline{y+z}-z^3$. Therefore, $q\times\overline{y+z}-y^3-3yz\times\overline{y+z}-z^3$ will be $=r$.

In this equation it is true indeed that the terms $q\times\overline{y+z}$ and $3yz\times\overline{y-z}$ have different ſigns. But they cannot be equal to each other: for, ſince the three terms y^3 and $3yz\times\overline{y-z}$ and z^3 are all marked with the ſign $-$, or are to be ſubtracted from the firſt term $q\times\overline{y+z}$, and the remainder is $=r$, it is evident, that $q\times\overline{y+z}$ muſt be greater than the ſum of all the three terms y^3, $3yz\times\overline{y+z}$, and z^3, taken together, and therefore, *à fortiori*, greater than $3yz\times\overline{y+z}$ alone. Therefore, no ſuch extermination of equal terms marked with contrary ſigns as took place in the transformed equations derived from the two former equations $x^3+qx=r$ and $x^3-qx=r$, can take place in this transformed equation derived from the equation $qx-x^3=r$ by ſubſtituting $y+z$ in its terms inſtead of x; and conſequently no ſuch method of reſolving the equation

tion $qx-x^3=r$ as has been found for the reſolution of the equations $x^3+qx=r$ and $x^3-qx=r$, can be obtained by means of that ſubſtitution.

Art. 17. Theſe are the methods of inveſtigation by which I conceive it to be probable, that CARDAN's rules for the reſolution of the cubic equations $x^3+qx=r$ and $x^3-qx=r$, together with the limitation of the rule relating to the latter of thoſe equations, and their inapplicability to the third equation $qx-x^3=r$, may have been diſcovered by the firſt inventor of them.